# The Absence of Fermionic Superradiance
# (A Simple Demonstration)

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## Abstract

Superradiant scattering, which can be thought of as the wave analogue of the Penrose process is revisited. As is well-known, boson fields display superradiance provided they have frequency in a certain range whereas fermion fields do not. A succinct superradiance-checking algorithm employing particle number or energy current is formally reviewed and then applied to the case of fermion field. The demonstrations of the absence of fermionic superradiance in terms of the particle number current exist in the literature but they are in the context of two-component SL(2,C) spinor formalism for massive spinor and SO(3,1) Dirac spinor formalism for massless spinor. Here we present an alternative demonstration in terms of both particle number and energy current but in a different context of local SO(3,1) Dirac spinor formalism for both massless and massive spinors. It appears that our presentation confirms the absence of fermionic superradiance in a more simple and systematic manner.

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## I. Introduction

A black hole is, by definition, a "region of no escape". No massive object or even the massless light ray, therefore, can ever be extracted from a black hole. When it comes to rotating black holes such as the Kerr family of solutions, however, things are not so simple and indeed energy can be extracted from black holes as was first noted by Penrose [1]. Briefly, this energy extraction mechanism proposed by Penrose and hence is called "Penrose process" [1,5] can be understood as follows. In Kerr geometry, the surface on which $g_{tt}$ vanishes does not coincide with the event horizon exept at the poles. The toroidal space inbetween the two surfaces is called "ergosphere" and in particular the outer boundary of this ergosphere on which $g_{tt}$ vanishes is dubbed "static limit" because it can be seen that inside of which no observer can possibly remain static. Namely the time translational Killing field $\xi^{\mu} = (\partial/\partial t)^{\mu}$ becomes spacelike inside the ergosphere and so does the conserved component $p_t$ of the four momentum. As a consequence, the energy of a particle in this ergoregion, as perceived by an observer at infinity, can be negative. This last fact leads to a peculiar possibility that, in principle, one can devise a physical process which extracts energy and angular momentum from the black hole. The Penrose process, however, requires a precisely timed breakup of the incident particle at the relativistic velocities and thus is not a very practical energy extraction scheme. Perhaps because of this reason, an alternative study of energy extraction mechanism, known as "superradiant scattering" [2,3,4,5] was considered. In a sense, it can be thought of as a wave analogue of the Penrose process. If a wave is incident upon a black hole, the part of the wave ("transmitted" wave) will be absorbed by the black hole and the part of the wave ("reflected" wave) will escape back to infinity. Normally, the transmitted wave will carry positive energy into the black hole and the reflected wave will have less energy than the incident wave. However, for a scalar wave with the time ($t$) and azimuthal angle ($\phi$) dependence given by $e^{i(m\phi-\omega t)}$ (with $m$ and $\omega$ being the azimuthal number and the frequency respectively), the transmitted wave will carry negative energy into the black hole and the reflected wave will escape to infinity with greater amplitude and energy than the originally incident one provided the scalar wave has the frequency in the range [2,4,5]

$$0 < \omega < m\Omega_H$$

where $\Omega_H$ denotes the angular velocity of the rotating hole at the event horizon. The "scalar waves" such as electromagnetic and gravitational waves exhibit this superradiance [2,4] when they have frequency in the range given above. Curiously enough, it is known that fermion fields do not display superradiance [3,5]. Conventionally, the typical way of demonstrating the presence or absence of superradiance is to define the reflection and transmission coefficients in terms of the solutions of the wave (field) equations in the background of the Kerr spacetime and then see if the reflection coefficient can exceed unity [4]. And unlike the case of scalar waves, the reflection coefficient in the case of fermion field never exceeds unity for all values of the frequency including the ones in the range given above [4]. In the present work, we would like to present a simple yet very solid formalism which demonstrates the absence of the superradiance in the case of massive or massless fermion field. It involves standard formulation of spinor field theory in curved background spacetime (the Kerr black hole background for our case) [6] associated with the Riemann-Cartan formulation of general relativity (which utilizes the soldering form particularly suitable for the case of spinor field). Thus in this formulation, we need to put the Kerr metric given in Boyer-Lindquist coordinates [7] in ADM's (3+1) space-plus-time split form to extract the soldering form (i.e., the non-coordinate basis 1-form). This, then, allows us to show that the energy current or the particle number current (which will be defined shortly in the next section) [5] of spinor field flowing into the black hole through the event horizon exactly vanishes establishing the absence of superradiance. The demonstration employing the fermionic particle number current may not be new [3]. But the existing works in the literature are all in the context of two-component SL(2,C) spinor formalism for massive spinor and SO(3,1) Dirac spinor formalism for massless spinor. Our formalism that we shall present here works with Dirac spinor (but in a different context from the existing work) associated with the local SO(3,1) group for both massless and massive spinors and it allows us to compute both the energy flux and the particle number flux in a more systematic and straightforward manner.

## II. General Formalism

In this section, we would like to first set up a general formalism [5] that allows us to determine whether or not the superradiance is actually present in the case of scalar or fermion field. Then later on we shall illustrate, as an exmple, the presence of superradiance in the case of scalar (complex scalar) field.

To this end, we introduce two quantities of central importance, "energy current" and "particle number current". First, we begin with the energy current. Generally, the "energy current" of a field in curved background spacetime is defined by [5]

$$J_{\mu} \equiv -T_{\mu\nu}\xi^{\nu} \tag{1}$$

with $\xi^{\mu} = (\partial/\partial t)^{\mu}$ being the time translational Killing field of a stationary, axisymmetric spacetime (which is the Kerr black hole spacetime for our case). This quantity is obviously conserved owing to the energy-momentum conservation and the Killing equation $\nabla^{\mu}\xi^{\nu} + \nabla^{\nu}\xi^{\mu} = 0$ satisfied by the Killing field $\xi^{\mu}$, i.e.,

$$\nabla^{\mu}J_{\mu} = -(\nabla^{\mu}T_{\mu\nu})\xi^{\nu} - T_{\mu\nu}(\nabla^{\mu}\xi^{\nu}) = 0.$$

Next, we turn to the particle number current. Generally speaking, for field theories with action possessing the global U(1) transformation (i.e., phase transformation) symmetry, (e.g. complex scalar field theory and fermion field theory) the associated Noether current can be identified with the particle number current. Namely the Noether current of the typical form

$$j^{\mu} = \frac{\delta\mathcal{L}}{\delta(\nabla_{\mu}\phi^{i})}\delta\phi^{i} \tag{2}$$

(where $\mathcal{L}$ denotes the Lagrangian density) is defined to be the particle number density. Then this particle number density is covariantly conserved as well due to the Euler-Lagrange's equation of motion and the invariance of the action, $\nabla_{\mu}j^{\mu} = 0$.

Now in order eventually to determine the presence or absence of the superradiant scattering, we consider a region $K$ of spacetime of which the boundary consists of two spacelike hypersurfaces $\Sigma_1$ at $(t)$ and $\Sigma_2$ at $(t + \delta t)$ (the constant time slice $\Sigma_2$ is a time translate of $\Sigma_1$ by

$\delta t$) and two timelike hypersurfaces $H$ (black hole horizon at $r = r_+$) and $S_\infty$ (large sphere at spatial infinity $r \to \infty$). The appropriate directions of the hypersurface normal vector $n^\mu$ on each part of the boundary are ; $n^\mu$ is future-directed on $\Sigma_1$, past-directed on $\Sigma_2$. It is pointing inward the black hole on the event horizon $H$ and pointing outward to infinity on $S_\infty$. Then next, consider integrating the quantity $\nabla^\mu J_\mu$ (which leads to the "energy flux" crossing each part of the boundary upon utilizing the Gauss's theorem) or $\nabla_\mu j^\mu$ (which leads to the "particle number current") over the region $K$ of spacetime. By using Gauss's theorem we have

$$\begin{aligned} 0 &= \int_K d^4x\sqrt{g}\nabla_\mu j^\mu \\ &= \int_{\partial K} d^3x\sqrt{h} n_\mu j^\mu \\ &= \int_{\Sigma_1(t)} n_\mu j^\mu + \int_{\Sigma_2(t+\delta t)} n_\mu j^\mu + \int_{H(r_+)} n_\mu j^\mu + \int_{S_\infty} n_\mu j^\mu \end{aligned} \tag{3}$$

where $h_{\mu\nu}$ denotes the 3-metric induced on the boundary $\partial K$ of the region $K$. Now the terms in the last line of eq.(3) need some explanations. For boson or fermion field with time dependence of the form $e^{-i\omega t}$ (which we shall assume throughout) the first two terms cancel with each other by time translation symmetry. The third term represents the net particle number flow into the rotating black hole while the last term stands for the net particle number flow out of $K$ to infinity, i.e., the outgoing minus incoming particle number through $S_\infty$ during the time $\delta t$. Thus we end up with the result

$$\int_{S_\infty} n_\mu j^\mu = -\int_{H(r_+)} n_\mu j^\mu \tag{4}$$

which states that the *net particle number flow out of $K$ or the "outgoing minus incoming particle number" equals "minus" of the net particle number flow into the rotating black hole.* Therefore, now we can establish the criterion for the occurrence of superradiant scattering ; If the quantity on the right hand side $\int_{H(r_+)} n_\mu j^\mu$, namely the net particle number flowing down the hole, is negative (zero or positive), it means that the outgoing particle number flux is greater (smaller) than the incident one and hence the superradiance is present (absent). Thus far we have established the criterion for the occurrence of superradiance in terms of

the "particle number current" $j^\mu$. An equivalent criterion can be derived in terms of the "energy current" $J_\mu$ if we replace $n_\mu j^\mu$ by $< n^\mu J_\mu >$ (where $< ... >$ denotes time averaged quantity) and replace "particle number current" with "energy current" respectively in the above formalism.

Also note that the hypersurface normal $n^\mu$ on the black hole event horizon $H$ is pointing inward the hole and hence is opposite to the direction of the Killing field [5]

$$\chi^\mu = \xi^\mu + \Omega_H \psi^\mu \tag{5}$$

which is outer normal to the rotating hole's event horizon (here $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial\phi)^\mu$ denotes Killing fields associated with the time translational and rotational isometries of the stationary, axisymmetric Kerr black hole spacetime respectively and $\Omega_H$ denotes the angular velocity of the event horizon of the hole).

Thus our task of checking the presence or absence of superradiance reduces to the computation of the net particle number (or energy) current flowing into the rotating hole through its event horizon

$$\int_{H(r_+)} n_\mu j^\mu = -\int_{H(r_+)} \chi_\mu j^\mu. \tag{6}$$

Now, before we demonstrate the absence of the superradiance in the case of fermion field in the next section, it will be instructive to illustrate the presence of the superradiance in a boson field case using the superradiance-checking formalism introduced above.

Thus consider a complex scalar field theory in a stationary, axisymmetric background spacetime (which we take to be the Kerr black hole geometry) described by the action [6] (here in this work, we employ the Misner-Thorne-Wheeler sign convention [8] in which the metric has the sign of $g_{\mu\nu} = diag(-+++)$)

$$S = -\int d^4x\sqrt{g}[\nabla_\mu \Phi^* \nabla^\mu \Phi + (M^2 + \xi R)\Phi^* \Phi] \tag{7}$$

and the classical field equations

$$\nabla_\mu \nabla^\mu \Phi - (M^2 + \xi R)\Phi = 0, \tag{8}$$

$$\nabla_\mu \nabla^\mu \Phi^* - (M^2 + \xi R)\Phi^* = 0$$

where $\xi$, $M$ and $R$ denotes some constant (for example, $\xi = 1/6$ with $M = 0$ corresponds to "conformal couplig"), the mass of the scalar field and the scalar curvature of the background spacetime respectively. Then we consider a situation when a complex scalar wave with particular frequency

$$\Phi(x) = \Phi_0(r, \theta)e^{i(m\phi - \omega t)} \tag{9}$$

is incident on and reflected by the Kerr black hole. Since the Lagrangian density of this complex scalar field in eq.(7) is invariant under the global U(1) (or phase) transformation

$$\Phi(x) \to e^{-i\alpha}\Phi(x),$$
$$\Phi^*(x) \to e^{i\alpha}\Phi^*(x)$$

corresponding Noether current exists and it is

$$j^\mu = -i(\Phi^* \nabla^\mu \Phi - \Phi \nabla^\mu \Phi^*). \tag{10}$$

This Noether current is the particle number current and it can be seen to be conserved owing to the classical field equations in eq.(8)

$$\nabla_\mu j^\mu = 0.$$

According to the criterion for the occurrence of the superradiance stated earlier, all we have to do now is to evaluate the net particle number flowing into the black hole, $\int_{H(r_+)} n_\mu j^\mu$ and see if it can be negative. Thus on the horizon $r = r_+$, we compute the particle number flux and it is

$$\begin{aligned} n_\mu j^\mu &= -\chi^\mu j_\mu \\ &= i(\Phi^* \chi^\mu \nabla_\mu \Phi - \Phi \chi^\mu \nabla_\mu \Phi^*) \\ &= i[\Phi^*(\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi})\Phi - \Phi(\frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi})\Phi^*] \\ &= 2(\omega - m\Omega_H) \mid \Phi_0 \mid^2 . \end{aligned} \tag{11}$$

Thus for a complex scalar field with frequency in the range

$$0 < \omega < m\Omega_H \tag{12}$$

the net particle number flowing down the hole is negative and hence

$$\int_{S_\infty} n_\mu j^\mu = -\int_{H(r_+)} n_\mu j^\mu > 0 \tag{13}$$

namely the outgoing minus incident particle number flux through the large sphere $S_\infty$ is positive indicating the occurrence of superradiance in the case of a scalar field. Before we end, we would like to comment on the following point. In the process of drawing the conclusion that the scalar field we considered displays superradiance provided it has the frequency in the range given above, almost no reference has been made to the specifics of the background spacetime geometry, i.e., the Kerr geometry. The only ingredient associated with the Kerr geometry entered the analysis was the use of the Killing field $\chi^\mu$ which is the outer normal to the rotating hole's event horizon. Thus we really need not know even the concrete form of the metric of Kerr spacetime. In a sense, this point may imply that the occurrence of superradiance in boson field case is an unshakeable, solid phenomenon but in another sense it makes us feel rather uncomfortable. In the case of fermion field we shall discuss in the following section, situation changes ; one needs the specifics of the Kerr geometry to reach the conclusion on the absence of superradiance, and we feel that this seems more natural.

## III. Absence of superradiance in the case of fermion field

As already mentioned and as we shall see shortly as well, in order to demonstrate the absence of superradiance in the fermion field case, one needs the concrete geometry structure of the background Kerr black hole. Besides, the standard formulation of spinor field theory in curved background spacetime is associated with the Riemann-Cartan formulation of general relativity in which one of the basic computational tools is the use of the non-holonomic basis 1-form (i.e., "soldering form"). Thus here we begin by casting the Kerr metric (given in Boyer-Lindquist coordinates [7]) into the ADM's (3+1) space-plus-time split form which proves to be suitable to be converted to the one in non-coordinate basis

$$
\begin{aligned}
ds^2 &= -N^2dt^2 + h_{rr}dr^2 + h_{\theta\theta}d\theta^2 + h_{\phi\phi}[d\phi + N^{\phi}dt]^2 \\
&= g_{\mu\nu}dx^{\mu}dx^{\nu} = \eta_{ab}e^{a}e^{b}
\end{aligned} \tag{14}
$$

where Greek indices refer to the accelerated frame of reference (i.e., coordinate basis, $\mu = t,\ r,\ \theta,\ \phi$) and the Roman indices refer to the locally inertial reference frame (i.e., non-coordinate basis, $a = 0,\ 1,\ 2,\ 3$). Also we used the definitions for the soldering form ("vierbein") $g_{\mu\nu} = \eta_{ab}e^{a}_{\mu}e^{b}_{\nu}$ and the non-coordinate basis 1-form $e^{a} = e^{a}_{\mu}dx^{\mu}$. In the ADM's (3+1) split form above, the lapse, shift functions and the spatial metric components are given respectively by

$$
\begin{aligned}
N^2(r,\theta) &= [\frac{\Delta - a^2\sin^2\theta}{\Sigma}] + R^{-2}(r,\theta)[\frac{r^2+a^2-\Delta}{\Sigma}]^2 a^2\sin^4\theta, \\
N^{\phi}(r,\theta) &= -R^{-2}(r,\theta)[\frac{r^2+a^2-\Delta}{\Sigma}]a\sin^2\theta, \quad N^{r} = N^{\theta} = 0, \\
h_{rr}(r,\theta) &= f^{-2}(r,\theta) = \frac{\Sigma}{\Delta}, \quad h_{\theta\theta}(r,\theta) = g^2(r,\theta) = \Sigma, \\
h_{\phi\phi}(r,\theta) &= R^2(r,\theta) = [\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}]\sin^2\theta, \\
h_{r\theta} &= h_{\theta r} = 0, \quad h_{r\phi} = h_{\phi r} = 0, \quad h_{\theta\phi} = h_{\phi\theta} = 0
\end{aligned} \tag{15}
$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2M_{KN}r + a^2$ with $M_{KN}$ denoting the mass of the Kerr black hole respectively. As is well known the event horizon developes at the larger zero of $\Delta(r_+) = 0$.

Actually, the virtue of writing the Kerr metric in the ADM's (3+1) split form as in eq.(14) above is that from which now one can read off the non-coordinate basis 1-form easily as follows

$$
\begin{aligned}
e^0 &= e^0_{\mu}dx^{\mu} = Ndt, \\
e^1 &= e^1_{\mu}dx^{\mu} = \sqrt{h_{rr}}dr = f^{-1}dr, \\
e^2 &= e^2_{\mu}dx^{\mu} = \sqrt{h_{\theta\theta}}d\theta = gd\theta \\
e^3 &= e^3_{\mu}dx^{\mu} = \sqrt{h_{\phi\phi}}(d\phi + N^{\phi}dt) = R(d\phi + N^{\phi}dt).
\end{aligned} \tag{16}
$$

Equvalently, the vierbein and the inverse vierbein can be read off as

$$e^a_\mu = \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & g & 0 \\ RN^\phi & 0 & 0 & R \end{pmatrix}, \quad e^\mu_a = \begin{pmatrix} N^{-1} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & g^{-1} & 0 \\ -N^{-1}N^\phi & 0 & 0 & R^{-1} \end{pmatrix} \tag{17}$$

Further, the spin connection 1-form $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ can be obtained from the Cartan's 1st structure equation, $de^a + \omega^a_b \wedge e^b = 0$ using the non-coodinate basis 1-form given in eq.(16). Here, however, we do not look for the spin connection since we shall not really need its explicit form in the discussion below leading to the conclusion on the absence of fermionic superradiance.

Now, for later use we write the $\gamma$-matrices with coordinate basis indices in accelerated frame (i.e., in Boyer-Lindquist coordinates) in terms of those with non-coordinate basis indices in locally-inertial frame using the soldering form (inverse vierbein) given above, i.e., $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$

$$\begin{aligned}
\gamma^t &= e^t_a\gamma^a = N^{-1}\gamma^0, \\
\gamma^r &= e^r_a\gamma^a = f\gamma^1, \\
\gamma^\theta &= e^\theta_a\gamma^a = g^{-1}\gamma^2, \\
\gamma^\phi &= e^\phi_a\gamma^a = -N^{-1}N^\phi\gamma^0 + R^{-1}\gamma^3.
\end{aligned} \tag{18}$$

With this preparation, now we consider the spinor field theory in the background of this Kerr spacetime described by the action [6]

$$S = \int d^4x\sqrt{g}\{\frac{i}{2}[\bar{\psi}\gamma^\mu\overrightarrow{\nabla}_\mu\psi - \bar{\psi}\gamma^\mu\overleftarrow{\nabla}_\mu\psi] - M\bar{\psi}\psi\} \tag{19}$$

and the classical field equations, i.e., curved spacetime Dirac equations

$$\begin{aligned}
&(i\gamma^\mu\overrightarrow{\nabla}_\mu - M)\psi = 0, \\
&\bar{\psi}(i\gamma^\nu\overleftarrow{\nabla}_\nu + M) = 0
\end{aligned} \tag{20}$$

where $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ is the curved spacetime $\gamma$-matrices obtained in eq.(18) above satisfying $\{\gamma^\mu(x),\ \gamma^\nu(x)\} = -2g^{\mu\nu}(x)$ with $e^a_\mu$ and $e^\mu_a$ being the vierbein and its inverse as obtained

in eq.(17) and they are defined by $g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)$ and $e^a_\mu e^\mu_b = \delta^a_b,\ e^\mu_a e^a_\nu = \delta^\mu_\nu$. And $\nabla_\mu = [\partial_\mu - \frac{i}{4}\omega^{ab}_\mu(x)\sigma_{ab}]$ is the covariant derivative with $\omega^{ab}_\mu(x)$ being the spin connection (that can, as mentioned, be obtained from the vierbein in eq.(17)) and $\sigma_{ab} = \frac{i}{2}[\gamma_a,\ \gamma_b]$ being the SO(3,1) group generator in the spinor representation.

Again, we consider a situation when a spinor wave with particular frequency

$$\psi(x) = u(p,s)\psi_0(r,\theta)e^{i(m\phi - \omega t)} \tag{21}$$

is incident on and reflected by the Kerr black hole. Here $s$ denotes spin state, $p^t = \omega$ and the 4-component Dirac spinor $u(p,s)$ satisfies the Dirac equation in curved spacetime given above. Since the Lagrangian density of this spinor field given in eq.(19) is also invariant under the global U(1) (or phase) transformation

$$\psi(x) \to e^{-i\alpha}\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x)e^{i\alpha}$$

corresponding Noether current exists and it is

$$\begin{aligned} j^\mu &= \frac{\delta\mathcal{L}}{\delta(\overrightarrow{\nabla}_\mu\psi)}\delta\psi + \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta(\bar{\psi}\overleftarrow{\nabla}_\mu)} \\ &= \bar{\psi}\gamma^\mu\psi. \end{aligned} \tag{22}$$

This Noether current is identified with the particle number current and it can be seen to be conserved due to the Dirac equations given above

$$\begin{aligned} \nabla_\mu j^\mu &= \bar{\psi}\gamma^\mu\overleftarrow{\nabla}_\mu\psi + \bar{\psi}\gamma^\mu\overrightarrow{\nabla}_\mu\psi \\ &= iM\bar{\psi}\psi - iM\bar{\psi}\psi = 0. \end{aligned} \tag{23}$$

An alternative way of deriving the number current and confirming its covariant conservation is to multiply two Dirac equations in eq.(20) by $\bar{\psi}$ and $\psi$ respectively and add them up to yield eq.(23). According to the criterion for checking the occurrence of the superradiance stated earlier in the previous section, all we have to do is to evaluate the net particle number

flowing into the black hole, $\int_{H(r_+)} n_\mu j^\mu$ and see if it can be negative. Thus on the horizon $r = r_+$, we compute the fermionic particle number flux and it is

$$
\begin{aligned}
n_\mu j^\mu &= -\chi^\mu j_\mu = -\bar{\psi} g_{\alpha\beta}\chi^\alpha \gamma^\beta \psi \\
&= -\bar{\psi} g_{\alpha\beta}(\delta^\alpha_t + \Omega_H \delta^\alpha_\phi)\gamma^\beta \psi \\
&= -\bar{\psi}[(g_{tt} + \Omega_H g_{t\phi})\gamma^t + (g_{t\phi} + \Omega_H g_{\phi\phi})\gamma^\phi]\psi \\
&= -\bar{\psi}[-N\gamma^0 + R(N^\phi + \Omega_H)\gamma^3]\psi = 0.
\end{aligned} \tag{24}
$$

where we used the relation between $\gamma$-matrices with coordinate basis indices in accelerated frame (i.e., in Boyer-Lindquist coordinates) and those with non-coordinate basis indices in locally-inertial frame, $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ derived earlier in eq.(18) and $g_{tt} = -[N^2 - R^2(N^\phi)^2]$, $g_{t\phi} = R^2 N^\phi$ and $g_{\phi\phi} = R^2$. And to get the last equality to zero we used

$$N^2(r_+,\ \theta) = 0, \quad N^\phi(r_+,\ \theta) = -(\frac{a}{r_+^2 + a^2}) = -\Omega_H.$$

This result indicates that the net fermionic particle number flux flowing down the hole through its event horizon is zero *irrespective of the frequency of the fermion field and regardless of its being massive or massless.* Therefore the outgoing minus incident fermionic particle number flux through the large sphere $S_\infty$ is zero

$$\int_{S_\infty} n_\mu j^\mu = -\int_{H(r_+)} n_\mu j^\mu = 0$$

establishing the absence of superradiance in the case of fermion field.

Thus far we have illustrated the absence of fermionic superradiance in terms of the particle number flux. One can draw the same conclusion in terms of the energy flux we introduced earlier by showing that it also is zero through the event horizon of the Kerr black hole. Thus in the following we shall do this.

As mentioned earlier, the energy current is defined to be

$$J_\mu = -T_{\mu\nu}\xi^\nu$$

with $\xi^\mu$ again being the time translational Killing field and this energy current is indeed conserved due to the energy-momentum conservation and the Killing equation satisfied by

$\xi^\mu$. For a fermion field described by the action given earlier, the energy-momentum tensor is given by [6]

$$
\begin{aligned}
T^{\mu\nu} &= \frac{1}{2(det\ e)}\eta^{ab}[e^\mu_a \frac{\delta S}{\delta e^b_\nu} + e^\nu_b \frac{\delta S}{\delta e^a_\mu}] \qquad (25)\\
&= \frac{i}{4}\{[\bar{\psi}\gamma^\mu \overrightarrow{\nabla}^\nu \psi - \bar{\psi}\gamma^\mu \overleftarrow{\nabla}^\nu \psi] + [\bar{\psi}\gamma^\nu \overrightarrow{\nabla}^\mu \psi - \bar{\psi}\gamma^\nu \overleftarrow{\nabla}^\mu \psi]\}.
\end{aligned}
$$

Now in exactly the same manner as we worked with the particle number current, we evaluate the net "time averaged" energy current flowing into the black hole, $\int_{H(r_+)} < n^\mu J_\mu >$ and see if it can be negative, but in this time using the form of the spinor wave with particular frequency given by

$$
\begin{aligned}
\psi(x) &= u(p,s)Re[\psi_0(r,\theta)e^{i(m\phi-\omega t)}], \qquad (26)\\
\bar{\psi}(x) &= \bar{u}(p,s)Re[\psi_0^*(r,\theta)e^{-i(m\phi-\omega t)}].
\end{aligned}
$$

Thus on the horizon $r = r_+$, we compute the fermionic time averaged energy flux and it is

$$
\begin{aligned}
< n^\mu J_\mu > &= - < \chi^\mu J_\mu > = < T_{\mu\nu}\chi^\mu \xi^\nu > \qquad (27)\\
&= < \frac{i}{4}\{[\bar{\psi}\chi^\mu \gamma_\mu \xi^\nu \overrightarrow{\nabla}_\nu \psi - \bar{\psi}\chi^\mu \gamma_\mu \xi^\nu \overleftarrow{\nabla}_\nu \psi] \\
&\quad + [\bar{\psi}\xi^\nu \gamma_\nu \chi^\mu \overrightarrow{\nabla}_\mu \psi - \bar{\psi}\xi^\nu \gamma_\nu \chi^\mu \overleftarrow{\nabla}_\mu \psi]\} >= 0
\end{aligned}
$$

where we used the facts that $\chi^\mu \gamma_\mu = 0$ on the event horizon as we have shown in eq.(24) above in the evaluation of $n_\mu j^\mu$ and that the two terms on the left hand side of the last line exactly cancel with each other since $\xi^\nu \gamma_\nu = (g_{tt}\gamma^t + g_{t\phi}\gamma^\phi) = [-N(r_+)\gamma^0 + R(r_+)N^\phi(r_+)\gamma^3] = R(r_+)N^\phi(r_+)\gamma^3$ and

$$
\begin{aligned}
\chi^\mu \overrightarrow{\nabla}_\mu \psi &= [(\omega - m\Omega_H)Y_0(r_+,\theta)\cos(m\phi - \omega t) - (\omega + m\Omega_H)X_0(r_+,\theta)\sin(m\phi - \omega t)]u(p,s) \\
&\quad - \frac{i}{4}(\omega^{ab}_t + \Omega_H \omega^{ab}_\phi)\sigma_{ab}\psi, \\
\bar{\psi}\chi^\mu \overleftarrow{\nabla}_\mu &= [(\omega - m\Omega_H)Y_0(r_+,\theta)\cos(m\phi - \omega t) - (\omega + m\Omega_H)X_0(r_+,\theta)\sin(m\phi - \omega t)]\bar{u}(p,s) \\
&\quad - \frac{i}{4}\bar{\psi}(\omega^{ab}_t + \Omega_H \omega^{ab}_\phi)\sigma_{ab}
\end{aligned}
$$

where $X_0(r,\theta) = Re[\psi_0(r,\theta)]$ and $Y_0(r,\theta) = Im[\psi_0(r,\theta)]$. Again, the net time averaged fermionic energy flux flowing down the hole through its event horizon is zero irrespective of

the frequency of the fermion field and regardless of its being massive or massless. Therefore the outgoing minus incident time averaged fermionic energy flux through the large sphere $S_\infty$ is zero confirming the absence of fermionic superradiance.

## IV. Discussions

Now we end with some comments. It is well-known that the behavior of both boson and fermion fields incident upon a rotating Kerr black hole is in close analogy to a famous effect in relativistic quantum mechanics known as the "Klein paradox". Recall that if a Klein-Gordon field in one spatial dimension is incident upon an electrostatic potential $V$, the reflected wave emerge with greater amplitude and energy than the incident one provided certain condition is met. For a Dirac field, however, the reflected wave will be smaller than the incident one. In quantum field theory, the interpretation of the Klein paradox is that in both the boson and fermion cases, particle-antiparticle pairs are spontaneously created in the strong electrostatic field associated with the potential V. And when incoming particles also are present, stimulated emision occurs in the boson case and in the classical limit, this results in the amplified reflected wave obtained with the classical analysis. The close analogy between the Klein paradox and the superradiant scattering of waves by a Kerr black hole suggests that spontaneous particle creation should occur near the Kerr black hole horizon. Indeed this is the case and it is directly related to the origin of Hawking evaporation of black holes [9,6].

Thus far the standard, traditional way of demonstrating the presence of superradiance in boson field case and the absence in fermion field case has been to define the reflection and transmission coefficients in terms of the solutions of the wave equations in the Kerr black hole background and see if the reflection coefficient can exceed unity under certain circumstances [4]. The demonstration in terms of the fermionic particle number current in the context of two-component SL(2,C) spinor formalism has been carried out as well [3]. Although we also employed the method for demonstrating the superradiance in terms of energy and particle number current, our presentation here is in different context from the existing ones namely, we work with SO(3,1) Dirac spinor and in terms of which the demonstration

appears to be very simple and more straightforward. In short, our presentation involves standard formulation of spinor field theory in curved background spacetime which makes use of Riemann-Cartan formulation of general relativity in which it is required to put the Kerr metric given in Boyer-Lindquist coordinates in ADM's (3+1) space-plus-time split form in order to extract soldering form. This, then, allowed us to show in a very straightforward manner that the energy current or the particle number current of fermion field flowing into the black hole through the event horizon exactly vanishes confirming the absence of superradiance. Finally, as we stressed earlier, it seems noteworthy that in the process of illustrating the presence of superradiance in the case of the scalar field, almost no reference has been made to the specifics of the background spacetime geometry, i.e., the Kerr geometry. As we have seen, we did not need to know even the concrete form of the metric of Kerr spacetime. Rather, the result, namely the condition for the occurrence of superradiance depends on the specifics of the scalar field itself, i.e., its time ($t$) and azimuthal angle ($\phi$) dependences. Thus it is interesting to remark on the contrast in the case of fermion field. In this time, one needs the specifics of the Kerr geometry to reach the conclusion on the absence of superradiance. And the result is completely independent of the specifics of the fermion field itself.

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